

# Killing forms on the five-dimensional Einstein-Sasaki $Y(p, q)$ spaces

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## Abstract

We present the complete set of Killing-Yano tensors on the five-dimensional Einstein-Sasaki  $Y(p, q)$  spaces. Two new Killing-Yano tensors are identified, associated with the complex volume form of the Calabi-Yau metric cone. The corresponding hidden symmetries are not anomalous and the geodesic equations are superintegrable.

Keywords: Einstein-Sasaki spaces, metric cone, Killing forms.

## 1 Introduction

In the last time Einstein-Sasaki geometries have become of large interest in connection with many modern studies in physics. In this paper we deal with the infinite family  $Y(p, q)$  of Einstein-Sasaki metrics on  $S^2 \times S^3$  [1, 2, 3, 4]. Such manifolds provide supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space  $Y(p, q)$  of an  $S^1$ -fibration over  $S^2 \times S^2$  with relative prime winding numbers  $p$  and  $q$  is topologically  $S^2 \times S^3$ .

In the present paper it will be shown that the equations of the geodesic motions on the  $Y(p, q)$  spaces are superintegrable. For this purpose we present the complete set of Killing-Yano tensors which play an essential role for the integrability of the equations of motion.

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In general a system could possess explicit and hidden symmetries encoded in the multitude of Killing vectors and higher rank Killing tensors respectively. The customary conserved quantities originate from symmetries of the configuration space of the system. They are represented by isometries of the metric generated by Killing vector fields. An extension of the Killing vector fields is given by the conformal Killing vector fields with flows preserving a given conformal class of metrics [5].

A natural generalization of Killing vector fields is represented by conformal Killing-Yano tensors. A conformal Killing-Yano tensor of rank  $p$  on a  $n$ -dimensional Riemannian manifold  $(M, g)$  is a  $p$ -form  $\omega$  which satisfies:

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner d\omega - \frac{1}{n-p+1} X^* \wedge d^* \omega, \quad (1)$$

for any vector field  $X$  on  $M$ . Here  $\nabla$  is the Levi-Civita connection of  $g$ ,  $X^*$  is the 1-form dual to the vector field  $X$  with respect to the metric  $g$ ,  $\lrcorner$  is the operator dual to the wedge product and  $d^*$  is the adjoint of the exterior derivative  $d$ . If  $\omega$  is co-closed in (1), then we obtain the definition of a Killing-Yano tensor [5]. Killing-Yano tensors are also called Yano tensors or Killing forms. A particular class of Killing forms is represented by the special Killing forms which satisfy, for some constant  $c$ , the additional equation [6]:

$$\nabla_X (d\omega) = c X^* \wedge \omega, \quad (2)$$

for any vector field  $X$  on  $M$ . Let us note that most known Killing forms are special.

Besides the antisymmetric generalizations of the Killing vectors one might also consider higher order symmetric tensors. A covariant symmetric field  $K$  of rank  $r$  on a Riemannian manifold  $(M, g)$  is a Stäckel-Killing tensor field if the symmetrization of its covariant derivative vanishes identically

$$\nabla_{(\lambda} K_{\mu_1, \dots, \mu_r)} = 0. \quad (3)$$

These symmetric tensors are associated with conserved quantities of degree  $r$  in the momentum variables, and generate symplectic transformations in the phase space of the system [7].

These two generalizations of the Killing vectors could be related. Given two rank  $r$  Killing forms  $\omega_{\mu_1 \dots \mu_r}$  and  $\sigma_{\mu_1 \dots \mu_r}$  it is possible to associate with them a Stäckel-Killing tensor of rank 2

$$K_{\mu\nu}^{(\omega, \sigma)} = (\omega_{\mu\lambda_2 \dots \lambda_r} \sigma_{\nu}^{\lambda_2 \dots \lambda_r} + \sigma_{\mu\lambda_2 \dots \lambda_r} \omega_{\nu}^{\lambda_2 \dots \lambda_r}). \quad (4)$$

At the quantum level the operator  $\nabla_\mu K^{\mu\nu} \nabla_\nu$  may be thought of as the equivalent of the classical conserved quantity  $K_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu$  which is constant along geodesics  $\gamma$ . The remarkable fact is that  $\nabla_\mu K^{\mu\nu} \nabla_\nu$  operating on scalar fields commutes with the Klein-Gordon operator  $\square = \nabla_\mu g^{\mu\nu} \nabla_\nu$  [8]. Therefore, when the Stäckel-Killing tensor  $K_{\mu\nu}$  is of the form (4), there are no quantum anomalies thanks to an integrability condition satisfied by the Killing-Yano tensors [8, 7, 9].

The organization of this paper is as follows: In the next Section we present what is essentially a brief review of how Einstein-Sasaki manifolds can be constructed as  $U(1)$  bundles over Einstein-Kähler manifolds. Further an Einstein-Sasaki metric may be defined as a  $(2n + 1)$ -dimensional Einstein metric such that the cone over it is a Kähler, Ricci-flat metric of complex dimension  $n + 1$ . In Section 3 we restrict to the five-dimensional  $Y(p, q)$  manifolds and present the complete set of Killing forms. Finally we give our conclusions in Section 4.

## 2 Progression from Einstein-Kähler to Einstein-Sasaki to Calabi-Yau manifolds

A Riemannian manifold  $(M_{2n+1}, g_S)$  of odd dimension  $2n + 1$  is Sasakian if and only if its metric cone [10]

$$C(M_{2n+1}) = \mathbb{R}_{>0} \times M_{2n+1}, \quad g_{\text{cone}} = dr^2 + r^2 g_S, \quad (5)$$

is Kähler with the complex dimension  $n + 1$ . Moreover a Sasakian metric  $g_{ES}$  is Einstein with  $\text{Ric} = 2n g_{ES}$  if and only if its metric cone is Ricci-flat and Kähler, i. e. a Calabi-Yau manifold.

On the other hand Sasakian manifolds can be constructed as principal  $S^1$ -bundle over a Kähler manifold  $M_{2n}$ . It is convenient to write the Einstein-Sasaki metric in the form [11, 12]

$$g_{ES} = g_{EK} + (d\alpha + \sigma)^2, \quad (6)$$

where  $g_{EK}$  is the metric of the Einstein-Kähler manifold  $M_{2n}$  with the complex dimension  $n$  and

$$\eta = d\alpha + \sigma, \quad (7)$$

is the Sasakian 1-form of the Einstein-Sasaki metric. The form  $\sigma$  satisfies

$$d\sigma = 2J_{EK}, \quad (8)$$

where  $J_{EK}$  is Kähler form of the Einstein-Kähler base manifold  $M_{2n}$ .

It is known that on Sasakian manifolds the Killing 1-form  $\eta$  together with the third rank form

$$\Psi = \eta \wedge d\eta, \quad (9)$$

are special Killing forms (2) with constants  $c = -2$  and  $c = -4$  respectively.

Furthermore, in [6] is given a complete description of compact Riemannian manifolds admitting special Killing forms. The special Killing forms are exactly those forms which translate into the parallel forms on the metric cone. Since the metric cone is either flat or irreducible, the problem of finding all special Killing forms is reduced to a holonomy problem [13]. In the case of the Einstein-Sasaki spaces the metric cone is Ricci-flat and Kähler of complex dimension  $n + 1$ , has holonomy  $SU(n + 1)$  and there are two additional Killing forms of degree  $n + 1$  besides (9). These two additional forms are given by the complex volume form of the Calabi-Yau manifold  $C(M_{2n+1})$  and its conjugate. To get real forms we evaluate the real, respectively the imaginary parts of the complex volume form.

### 3 $Y(p, q)$ manifolds

The starting point is the explicit local metric of the 5-dimensional  $Y(p, q)$  manifold given by the line element [1, 2, 14]

$$ds_{ES}^2 = \frac{1 - cy}{6}(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}(d\psi - \cos \theta d\phi)^2 + w(y) \left[ d\alpha + \frac{ac - 2y + cy^2}{6(a - y^2)}[d\psi - \cos \theta d\phi] \right]^2, \quad (10)$$

where

$$w(y) = \frac{2(a - y^2)}{1 - cy}, \quad q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2}, \quad (11)$$

and  $a, c$  are constants. A detailed discussion of the range of these parameters is given in [2] in connection with the regularity properties of the  $Y(p, q)$  metrics.

The coordinate change  $\alpha = -\frac{1}{6}\beta - \frac{1}{6}c\psi'$ ,  $\psi = \psi'$  takes the line element (10) to the following form

$$ds_{ES}^2 = \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{p(y)}dy^2 + \frac{1}{36}p(y)(d\beta + c\cos\theta d\phi)^2 + \frac{1}{9}[d\psi' - \cos\theta d\phi + y(d\beta + c\cos\theta d\phi)]^2, \quad (12)$$

with  $p(y) = w(y)q(y)$ .

The Sasakian 1-form of the  $Y(p, q)$  space is

$$\eta = \frac{1}{3}d\psi' + \sigma, \quad (13)$$

with

$$\sigma = \frac{1}{3}[-\cos\theta d\phi + y(d\beta + c\cos\theta d\phi)], \quad (14)$$

connected with local Kähler form  $J_{EK}$  as in (8).

The form of the metric (10) with the 1-form (13) is the standard one for a locally Einstein-Sasaki metric with  $\frac{\partial}{\partial\psi'}$  the Reeb vector field.

From the isometries  $SU(2) \times U(1) \times U(1)$  the momenta  $P_\phi, P_\psi, P_\alpha$  and the Hamiltonian describing the geodesic motions are conserved [15, 14].  $P_\phi$  is the third component of the  $SU(2)$  angular momentum, while  $P_\psi$  and  $P_\alpha$  are associated with the  $U(1)$  factors. Additionally, the total  $SU(2)$  angular momentum given by

$$J^2 = P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2 + P_\Psi^2, \quad (15)$$

is also conserved.

In what follows we are looking for further conserved quantities specific for motions in Einstein-Sasaki spaces. First of all, on the Sasakian manifold with the 1-form (7), the form (9)

$$\begin{aligned} \Psi = \frac{1}{9}[(1-cy)\sin\theta d\theta \wedge d\phi \wedge d\psi' + dy \wedge d\beta \wedge d\psi' \\ + c\cos\theta dy \wedge d\phi \wedge d\psi' - \cos\theta dy \wedge d\beta \wedge d\phi \\ + (1-cy)y\sin\theta d\beta \wedge d\theta \wedge d\phi], \end{aligned} \quad (16)$$

is a special Killing form (2). Let us note also that

$$\Psi_k = (d\eta)^k, \quad k = 1, 2, \quad (17)$$

are closed conformal Killing forms (also called  $\star$ -Killing forms).

On the Einstein-Kähler manifold  $M_{2n}$  we introduce the complex vierbeins  $\zeta_1$  and  $\zeta_2$  such that

$$\begin{aligned}\zeta_1 &= \frac{\sqrt{1-cy}}{\sqrt{6}} d\theta + i \frac{\sqrt{1-cy}}{\sqrt{6}} \sin \theta d\phi, \\ \zeta_2 &= \frac{dy}{\sqrt{p(y)}} + i \frac{\sqrt{p(y)}}{6} (d\beta + c \cos \theta d\phi),\end{aligned}\tag{18}$$

and its Kähler form (8) is

$$J_{EK} = -i(\zeta_1 \wedge \bar{\zeta}_1 + \zeta_2 \wedge \bar{\zeta}_2).\tag{19}$$

On the Calabi-Yau manifold  $C(M_{2n+2})$  the Kähler form is

$$J_{cone} = r dr \wedge \eta + r^2 J_{EK}.\tag{20}$$

The complex vierbeins on the metric cone are chosen such that  $\Lambda_\mu = r\zeta_\mu$  for  $\mu = 1, 2$  and

$$\begin{aligned}\Lambda_3 &= \frac{dr}{r} + i\eta \\ &= \frac{dr}{r} + i \frac{1}{3} [d\psi' - \cos \theta d\phi + y(d\beta + c \cos \theta d\phi)].\end{aligned}\tag{21}$$

Now the complex volume form on the Calabi-Yau manifold is [16, 17]

$$dV = \Lambda_1 \wedge \Lambda_2 \wedge \Lambda_3.\tag{22}$$

Finally, let us note also that the Einstein-Sasaki  $Y(p, q)$  manifold is identified with the submanifold  $\{r = 1\}$  of the Calabi-Yau cone manifold. Accordingly, the additional Killing 3-forms of the  $Y(p, q)$  spaces are inferred from the real and imaginary part of the complex volume form (22):

$$\begin{aligned}\Xi &= \text{Re } dV|_{r=1} \\ &= -\frac{\sqrt{1-cy}}{3\sqrt{6}\sqrt{p(y)}} \left[ \sin \theta d\phi \wedge dy \wedge d\psi' + y \sin \theta d\phi \wedge dy \wedge d\beta \right. \\ &\quad \left. + \frac{p(y)}{6} d\theta \wedge d\beta \wedge d\psi' + \frac{p(y)}{6} c \cos \theta d\theta \wedge d\phi \wedge d\psi' \right. \\ &\quad \left. - \frac{p(y)}{6} \cos \theta d\theta \wedge d\beta \wedge d\phi \right],\end{aligned}\tag{23}$$

$$\begin{aligned}
\Upsilon &= \text{Im } dV|_{r=1} \\
&= \frac{\sqrt{1-cy}}{3\sqrt{6}\sqrt{p(y)}} \left[ d\theta \wedge dy \wedge d\psi' - \cos\theta d\theta \wedge dy \wedge d\phi \right. \\
&\quad \left. + y d\theta \wedge dy \wedge d\beta + c y \cos\theta d\theta \wedge dy \wedge d\phi \right. \\
&\quad \left. - \frac{p(y)}{6} \sin\theta d\phi \wedge d\beta \wedge d\psi' \right]. \tag{24}
\end{aligned}$$

The Stäckel-Killing tensors associated with the Killing forms  $\Psi, \Xi, \Upsilon$  are constructed as in (4). The list of the non vanishing components of these Stäckel-Killing tensors is quite long and will be given elsewhere. Together with the Killing vectors  $P_\phi, P_\psi, P_\alpha$  and the total angular momentum  $J^2$  (15) these Stäckel-Killing tensors provide the superintegrability of the  $Y(p, q)$  geometries.

## 4 Conclusions

In this paper we have presented the complete set of Killing forms on five-dimensional Einstein-Sasaki  $Y(p, q)$  spaces. The multitude of Stäckel-Killing tensors associated with these Killing forms implies the superintegrability of the geodesic motions.

In connection with the third rank Killing-Yano tensors on the  $Y(p, q)$  spaces let us note an interesting geometrical interpretation of the Lax representation [18, 19, 20].

In the theory of the classical spinning particles additional non-generic supersymmetries are generated from Killing-Yano tensors. At the quantum level from Killing forms one can construct Carter-McLenaghan like operators [21] which commute with the standard Dirac operator. It is also worth noting that in the full quantum theory Yano symmetries are not anomalous [7].

These remarkable properties of the Killing forms offer new perspectives in the investigation of the supersymmetries, separability of Hamilton-Jacobi, Klein-Gordon, Dirac equations on  $Y(p, q)$  spaces.

## Acknowledgments

Support through CNCSIS program PN-II-ID-PCE-2011-3-0137 is acknowledged.

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